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Symmetries on Partially Ordered Abelian Groups

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Recently, Foulis (Foulis, D. J. (2003). Compressible groups, *Mathematica Slovaca* **53**, 433–455.) characterized compressions on effect rings, which were introduced as a generalization of unital C*-algebras in the context of ordered abelian groups with order units. In the present paper, we characterize a class of symmetries on effect rings and show their relations to compressions. This characterization leads to a generalization of the notion of orthosymmetric orthoposets (Mayet, R., Pulmannová, S. (1994). Nearly orthosymmetric ortholattices and Hilbert spaces, *Foundations of Physics* **24**, 1425–1437.) to symmetric ordered abelian groups with order units.

KEY WORDS: effect ring; projection; symmetry; compression; symmetric group.

1. INTRODUCTION

This paper was inspired by the pioneering work of David Foulis. In (Foulis, 2003), he initiated the study of compressions on partially ordered abelian groups with order units. This study has been inspired by the important notion of compressions and dilations on $\mathcal{B}(H)$, the set of bounded operators on a Hilbert space H (Riesz and Nagy, 1955). We recall that if $P = P^2 = P^*$ is a projection operator on H, then the mapping $J_P\colon \mathcal{B}(H) \to \mathcal{B}(H)$ defined by $J_P(A) := PAP$, $A \in \mathcal{B}(H)$, is the compression determined by P, and conversely, if $D \in \mathcal{B}(H)$ and $A = J_P(D) = PDP$, then D is called the dilation of A. There are various dilation theorems (Riesz and Nagy, 1955; Paulsen, 2002), that characterize various classes of maps into $\mathcal{B}(H)$ as compressions to H of "nicer" maps into $\mathcal{B}(K)$, where K is a Hilbert space containing H. For instance, using the well-known Naimark dilation theorem, a positive operator valued measure can be dilated to a projection valued measure.

If P is a projection on H, then the operator $S := 2P - I$ is a symmetry, that is, a selfadjoint unitary operator on H, and there is a one-to-one correspondence between symmetries and projections. Moreover, if $S = 2P - I$

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is a symmetry, then the mapping U_S: $\mathcal{B}(H) \to \mathcal{B}(H)$ defined by $U_S(A) = SAS$ is an automorphism of $\mathcal{B}(H)$ such that $U_S \circ U_S$ is the identity. Such an automorphism is also called a symmetry. Owing to the one-to-one correspondence between symmetries and projections, every symmetry is generated by a projection. This suggests a close connection between compressions and symmetries. Symmetries play an important role in the foundations of quantum mechanics, see e.g., (Wigner, 1959).

Symmetries in the context of orthomodular lattices were introduced by Mayet (1992), and in the context of orthomodular posets in Mayet and Pulmannová, (1994). The notions of orthosymmetric ortholattices (OSOL), respectively orthosymmetric orthoposets (OSOP) were introduced as an ortholattice (orthoposet) in which to every element an automorphism is assigned such that a few natural axioms hold. It was shown that in the ortholattice (orthoposet) the orthomodular law is necessarily satisfied.

In this paper, we introduce the notion of a symmetry of a partially ordered abelian group G with order unit u. We define symmetries generated by elements in the unit interval $[0, u]$, and show that that the generators must be principal elements. In analogy with orthosymmetric orthoposets, we introduce the notion of a symmetric group, and investigate its properties. It is shown that if G is generated by an archimedean effect ring $\mathcal{R}$ with generative unit 1, in which $\mathcal{R}^+$ consists of squares, then G is a symmetric group, and the generators of symmetries coincide with projections. In addition, if for a projection p, every $g \in G$ can be decomposed into a part g_s^p which is stable under the symmetry U_p generated by p, and a part g_a^p which changes the sign under the action of U_p, then U_p is of the form $U_p(g) = (2p - 1)g(2p - 1)$, $g \in G$. Applying results from (Foulis, 2003), connections between symmetries and compressions on archimedean effect rings are found.

2. BASIC DEFINITIONS

A partially ordered group is an additively written abelian group G with a positive cone G^+, which induces a translation invariant partial order $\leq$ on G according to $g \leq h \Leftrightarrow h - g \in G^+$. If H is a subgroup of G, then H forms a partially ordered group with the positive cone $H^+ := H \cap G^+$. An element $u \in G^+$ is called an *order unit* iff for every $g \in G$ there is a positive integer n such that $-nu \leq g \leq nu$. If $G = G^+ - G^+$, then G is said to be *directed*. If G has an order unit, then G is directed. An element $u \in G^+$ is *generative* if every element $g \in G^+$ can be written as a finite sum $g = g_1 + \ldots + g_n$ with $0 \leq g_i \leq u$, $i = 1, 2, \ldots, n$. If G is directed, then a generative element in G^+ is automatically an order unit in G.

An *effect algebra* is a partial algebra $(E; \oplus, 0, 1)$ with a binary partial operation $\oplus$ and two nullary operations 0, 1 satisfying the following conditions (Bennett and Foulis, 1997).

(E1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.
(E2) If $a \oplus b$ and $(a \oplus b) \oplus c$ are defined, then $b \oplus c$ and $a \oplus (b \oplus c)$ are defined and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.
(E3) For every $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$.
(E4) If $a \oplus 1$ exists, then $a = 0$

Let E be an effect algebra. We say that $a, b \in E$ are *orthogonal*, written $a \perp b$, iff $a \oplus b$ is defined. A partial order can be defined on E by setting $a \leq b$ iff there is $c \in E$ such that $a \perp c$ and $a \oplus c = b$. It turns out that if such c exists, it is unique. We put $c = b \ominus a$. In this ordering, 0 is the smallest and 1 is the largest element in E. We also have $a \perp b$ iff $a \leq b'$.

Recall that an *orthoposet* is a partially ordered set P with the first and last elements 0 and 1, respectively, endowed with a unary operation (orthocomplementation) $a \mapsto a'$ such that $a \leq b \Rightarrow b' \leq a'$, $a' = (a')' = a$, $a \vee a' = 1$, and $a \vee b \in P$ whenever $a \leq b'$, i.e., $a \perp b$, for all elements $a, b \in P$. An orthoposet becomes an orthomodular poset (OMP) iff the orthomodular law is satisfied, i.e., $a \leq b \Rightarrow b = a \vee (a' \wedge b)$. An effect algebra becomes an OMP iff

$$a \oplus b = a \vee b \ \text{ whenever } a \perp b.$$

For more details on orthomodular posets see (Pták and Pulmannová, 1991), on effect algebras (Dvurečenskij and Pulmannová, 2000).

Let G be a partially ordered abelian group with generative element $u \in G^+$. The interval $E{:} = [0, u] \subset G^+$ can be endowed with a structure of an effect algebra if we define $a, b \in E$, $a \perp b$ iff $a + b \in E$ and then $a \oplus b = a + b$, and $a' = u - a$. An element $p \in E$ is called *sharp* if $p \wedge (u - p) = 0$ (the glb taken in E), and p is called *principal* if $x, y \leq p$, $x \perp y$ imply $x + y \leq p$. A principal element is sharp. Indeed, assume $x \leq p$, $x \leq p'$, then $x + p \leq p \Rightarrow x = 0$. An element $p \in E$ is *central* iff p and p' are principal and every $x \in E$ can be uniquely written as $x = x_1 + x_2$, $x_1 \leq p$, $x_2 \leq p'$ (Greechie *et al.*, 1995).

3. SYMMETRIES OF PARTIALLY ORDERED ABELIAN GROUPS

Definition 3.1. Let G be a partially ordered abelian group with a generating element $u \in G^+$. A mapping $U{:}\ G \to G$ is called a *symmetry* if

(i) U is an order automorphism of G,
(ii) $U(u) = u$,
(iii) $U \circ U = id_G$.

Clearly, for any symmetry U, $U(E) = E$. Let $U\colon G \to G$ be a symmetry. Let

$$S_U\colon = \{g \in G : U(g) = g\} \tag{1}$$

be the stable part of $U(G)$. Then S_U is a subgroup of G (not necessarily directed nor convex). Clearly $u \in S_U \cap E$ and $S_U \cap E$ forms a sub-effect algebra of E.

Definition 3.2. We say that a symmetry U_p of G is *generated* by an element $p \in E$ iff the following conditions hold:

(i) $[0, p] \cup [0, p'] \subset S_{U_p}$,
(ii) $S_{U_p} \cap E \simeq [0, p] \times [0, p']$, i.e., p is central in $S_{U_p} \cap E$.

If p generates U_p, we say that p is a *generator* of U_p

Lemma 3.3. *If $p \in E$ generates a symmetry U_p, then p is a principal element in E.*

Proof: Let $0 \le y, z \le p$ and $y + z \le u$. By (i) of Definition 3.2., $y, z \in S_{U_p} \cap E$, and since by (ii) of Definition 3.2 , p is central in $S_{U_p} \cap E$, we have $y + z \le p$. □

Observe that if $S_p\colon = S_{U_p}$, $S_{p'}\colon = S_{U_{p'}}$, then $S_p \cap E = S_{p'} \cap E$. Consequently, if a symmetry U_p is uniquely defined by its generator, then $U_p = U_{p'}$. Also, for the symmetry U_u generated by u, we have $S_{U_u} \cap E = E$, which implies $U_u = U_0 = id_G$.

Definition 3.4 Let G be a partially ordered abelian group with generative element $u \in G^+$. Assume that the set $P(E)$ of principal elements of E forms an orthoposet. We say that G is a *symmetric group* if there is a binary operation

$$U : P(E) \times G \to G$$

such that (with $U(p, g)$ denoted by $U_p(g)$) the following holds:

(i) $\forall p \in P(E)$, $U_p\colon G \to G$ is a symmetry generated by p,
(ii) $\forall p, q \in P(E)$, $U_p \circ U_q = U_{U_p(q)} \circ U_p$,
(iii) $p, q \in P(E)$, $p \perp q \;\Rightarrow\; U_p \circ U_q = U_{p+q}$.

Acccording to Mayet and Pulmannová (1994), the notion of a *symmetric orthoposet* (OSOP, in short) can be introduced as follows. An OSOP is an orthoposet P endowed with a binary operation U, satisfying the following axioms, where $U(a, b)$ is denoted by $U_a(b)$:

(S1) For every $a \in P$, $U_a \colon P \to P$ is an automorphism of the orthoposet P.
(S2) For every $a, b \in P$, $U_a(b) = b$ iff $b = b_1 \vee b_2, b_1 \leq a, b_2 \leq a'$.
(S3) $U_a \circ U_b = U_{U_a(b)} \circ U_a$ and $U_a \circ U_a = id_P$.
(S4) $a \perp b$ implies $U_{a \vee b} = U_a \circ U_b$.

Lemma 3.5. *If P is an OSOP, then P is an orthomodular poset (OMP).*

Proof: By (S4), $U_a \circ U_{a'} = U_{a \vee a'} = U_1 = id_P$, which implies, by (S3), that $U_{a'} = U_a$. Let $U_a(b) = b$, then by (S2), $b = b_1 \vee b_2$ with $b_1 \leq a$, $b_2 \leq a'$. Then by (S4), $U_b(a) = U_{b_1 \vee b_2}(a) = U_{b_1} \circ U_{b_2}(a)$. Now $b_2 \leq a'$ implies $a \leq b_2'$, hence $U_{b_2}(a) = U_{b_2'}(a) = a$. Similarly, $b_1 \leq a$ implies $a' \leq b_1'$, hence $U_{b_1}(a') = a'$, and since U_{b_1} is an automorphism, $U_{b_1}(a) = a$. Therefore $U_b(a) = a$, which means by (S2) that $a = a_1 \vee a_2, a_1 \leq b, a_2 \leq b'$. We proved that for $a, b \in P$, $b = b_1 \vee b_2$ with $b_1 \leq a, b_2 \leq a'$ implies $a = a_1 \vee a_2, a_1 \leq b, a_2 \leq b'$. By (Beran, 1984), this is equivalent to orthomodularity. □

Proposition 3.6. *Let G be a partially ordered abelian group with generative element $u \in G^+$ such that $P(E)$ forms an orthoposet. If G is symmetric group then $P(E)$ is an orthosymmetric orthoposet (OSOP) in the sense of* Mayet and Pulmannová (1994). *In particular, $P(E)$ is an orthomodular poset.*

Proof: Assume that G is a symmetric group. To prove (S1), S(3) and (S4), it suffices to prove that U restricted to $P(E) \times P(E)$ gives values in $P(E)$.

Let $p, q \in P(E)$ and let $0 \leq x, y \leq U_p(q)$ be such that $x + y \leq u$. Then $U_p(x), U_p(y) \leq q$, $U_p(x) + U_p(y) = U_p(x + y) \leq u$, and since q is principal, $U_p(x) + U_p(y) \leq q$. Applying U_p to both sides, we get $x + y \leq U_p(q)$. This proves that $U_p(q) \in P(E)$. Properties (i),(ii) and (iii) restricted to $P(E)$ imply (S1), (S3) and (S4).

To show (S2), let $p, q \in P(E)$ be such that $U_p(q) = q$. Then $q = e_1 + e_2$, $e_1, e_2 \in E, e_1 \leq p,\ e_2 \leq p'$. Since $p, q \in E \cap S_p$, and p is central in $E \cap S_p$, we have $q = q \wedge p + q \wedge p'$, where the infimum is taken in $E \cap S_p$. As for any $f \in E$, $f \leq p, q$, we have $f \in S_p$, $q \wedge p$ is the infimum of p, q also in E. Similarly $q \wedge p'$ is the infimum of p' and q in E. From $e_1 + e_2 = q = q \wedge p + q \wedge p'$ and $e_1 \leq q \wedge p,\ e_2 \leq q \wedge p'$, we obtain $e_1 = q \wedge p, e_2 = q \wedge p'$.

Assume that $x, y \in E, x, y \leq q \wedge p$, and $x + y \leq u$. Then $x, y \leq p, x, y \leq q$, and since p, q are principal, $x + y \leq p, q$, hence $x + y \leq p \wedge q$. It follows that $p \wedge q$ is principal, and $q = p \wedge q + p' \wedge q$ with $p \wedge q \leq p$, $q \wedge p' \leq p'$, $p \wedge q, p' \wedge q \in P(E)$.

Conversely, if $q \in P(E)$ is such that $q = q_1 + q_2$ with $q_1 \leq p$, $q_2 \leq p'$, $q_1, q_2 \in P(E)$, then by (ii) of Definition 3.2 , $U_p(q) = U_p(q_1) + U_p(q_2) = q_1 + q_2 = q$.

Notice that by Lemma 3.5 , $P(E)$ is an OMP. □

4. EFFECT RINGS

Definition 4.1. (Greechie *et al.*, 1995; Foulis, 2004) An *effect ring* is a ring $\mathcal{R}$ with a unit 1 such that $(\mathcal{R}, +)$ forms a partially ordered abelian group under addition with positive cone $\mathcal{R}^+$, and the following conditions hold for all $a, b \in \mathcal{R}^+$:

(i) $ab = ba \Rightarrow ab \in \mathcal{R}^+$.
(ii) $aba \in \mathcal{R}^+$.
(iii) $aba = 0 \Rightarrow ab = ba = 0$.
(iv) $(a - b)^2 \in \mathcal{R}^+$.
(v) $1 \in \mathcal{R}^+$.

Define $G(\mathcal{R}) := \mathcal{R}^+ - \mathcal{R}^+$ to be the additive group generated by $\mathcal{R}^+$ and understand that $G(\mathcal{R})$ is the directed partially ordered abelian group with positive cone $G(\mathcal{R})^+ := \mathcal{R}^+$. Define $E := [0, 1] \subset \mathcal{R}^+$.

Example 4.2. Let $\mathcal{R}$ be a unital C*-algebra, $\mathcal{R}^+ := \{aa^* : a \in \mathcal{R}\}$. $\mathcal{R}$ is an effect ring, $G(\mathcal{R})$ is the additive group of selfadjoint elements of $\mathcal{R}$, and 1 is a generative order unit in $G(\mathcal{R})$.

An element p in an effect ring $\mathcal{R}$ is called a *projection* if $p \in \mathcal{R}^+$ and $p = p^2$. The set of all projections in $\mathcal{R}$ is denoted by $P(\mathcal{R})$. If $p \in P(\mathcal{R})$, then $p' := 1 - p \in P(\mathcal{R})$. (indeed, by (iv) and (v) of the above definition, $(1 - p)^2 = 1 - p \in \mathcal{R}^+$). Observe that $1 - p \in \mathcal{R}^+$ implies $p \le 1$, hence $P(\mathcal{R}) \subset E$. The set $P(\mathcal{R})$ forms an orthomodular poset (OMP)(Greechie *et al.*, 1995).

Lemma 4.3. *In an effect ring $\mathcal{R}$ the following holds:*

(i) *For $x \in E, x \le p, p \in P(\mathcal{R})$ iff $pxp = px = xp = x$.*
(ii) *$p \in P(\mathcal{R})$ iff p is principal in E iff p is sharp in E.*

Proof: (i) Observe that if $p \le x \le 1$, then $p \le pxp \le p$ by (ii) of Definition 4.1, which implies $p = pxp$, whence $px = xp = pxp = p$. If $x \le p$, then $p' \le x'$, $p' = 1 - p \Rightarrow (1 - p) = (1 - p)(1 - x) = 1 - x - p + px \Rightarrow x = px$. Similarly we obtain $x = xp$. Hence $x = xp = pxp = px$. Conversely, if $x = pxp = xp = px$, then $0 \le p(1 - x)p = p - pxp = p - x$, hence $x \le p$.

(ii) Assume $p = p^2$. If $0 \le x, y \le p$ and $x + y \le 1$, then by part (i), $x = pxp$, $y = pyp$, hence $x + y = pxp + pyp = p(x + y)p \le p$. This proves that a projection is principal, hence sharp in E.

Observe that $0 \le p, 1 - p$ and $p(1 - p) = (1 - p)p$ implies by (i) of Definition 4.1 that $p(1 - p) \ge 0$. If p is sharp, then $p \wedge (1 - p) = 0$, and since $p(1 - p) = (1 - p)p = p - p^2 = (1 - p) - (1 - p)^2$, we have $p - p^2 \le p, 1 - p$, so that $p = p^2$. □

Recall that a partially ordered abelian group G is said to be *archimedean* if whenever $x, y \in G$ are such that $nx \le y$ for all positive integers n, then $x \le 0$, and G is said to be *unperforated* if $nx \ge 0$ for some positive integer n implies $x \ge 0$. Note that any unperforated partially ordered abelian group must be torsion-free as an abelian group. Notice that if $\mathcal{R}$ is archimedean, then $G(\mathcal{R})$ is archimedean, too. According (Goodearl, 1986), $G(\mathcal{R})$ is unperforated.

Theorem 4.4. *Let $\mathcal{R}$ be an archimedean effect ring with no element $x \neq 0$ with $x^2 = 0$, and with $\mathcal{R}^+ := \{a^2 : a \in G(\mathcal{R})\}$. Define, for $p \in P(\mathcal{R})$,*

$$U_p(x) := (2p-1)x(2p-1), \; x \in G(\mathcal{R}). \tag{2}$$

Then U_p is a symmetry of $G(\mathcal{R})$ generated by p. In addition, the following condition is fufilled:

(AS) $\forall g \in G(\mathcal{R}), g = g_s + g_a, U_p(g_s) = g_s, U_p(g_a) = -g_a.$

Proof: We have to check conditions (i) – (iii) of Definition 3.1.

(i) Additivity of U_p is clear.

$$\begin{aligned} U_p(x) = 0 \; &\Leftrightarrow \; (2p-1)x(2p-1) = 0 \\ &\Leftrightarrow U_p(0) = x \; \Leftrightarrow \; x = 0. \end{aligned}$$

(ii) $(2p-1)1(2p-1) = 4p - 2p - 2p + 1 = 1.$

(iii) $U_p \circ U_p(x) = (2p-1)((2p-1)x(2p-1))(2p-1) = x.$

This implies that U_p is injective. From $U_p \circ U_p(x) = x$ (by (iii)), we conclude that U_p is surjective. Hence U_p is a group automorphism.

Let $g \in G(\mathcal{R})^+$, then $g = x^2$ for some $x \in G(\mathcal{R})$. We have

$$\begin{aligned} U_p(g) &= (2p-1)x^2(2p-1) = (2p-1)x(2p-1)(2p-1)x(2p-1) \\ &= U_p(x)^2 \geq 0. \end{aligned}$$

We proved so far that U_p is an order preserving automorphism of $G(\mathcal{R})$. It is easy to see that $U_p(G(\mathcal{R})^+) = G(\mathcal{R})^+$.

Now we have to check conditions of Definition 3.2.

(i) Let $0 \leq x \leq p$. By Lemma 3.3 (i), $x = px = xp = pxp$. Therefore $(2p-1)x(2p-1) = x$. Similarly, if $0 \leq x \leq p'$, then $xp' = p'x = x$, so that $(2p'-1)x(2p'-1) = x$, and since $2p' - 1 = 1 - 2p$, we get $U_p(x) = x$.

(ii) Assume $x \in E$, $x = x_1 + x_2$ with $x_1 \leq p$, $x_2 \leq p'$. Then $U_p(x) = U_p(x_1) + U_p(x_2) = x_1 + x_2 = x$. Conversely, let $x \in E$ be such that $U_p(x) = x$. This yields $x = (2p-1)x(2p-1) = 4pxp - 2xp - 2px + x$, which entails $4pxp - 2xp - 2px = 0$. Multiplying the last equality by p from the right and from the left, and using the fact that $G(\mathcal{R})$ is torsio-free, we obtain, respectively, $pxp = xp$ and $pxp = px$, so $xp = px = pxp$. We can write

$$x = pxp + p'xp' + p'xp + pxp',$$

and owing to $px = xp$, the latter two terms equal 0. Therefore

$$x = pxp + p'xp', \ pxp \le p, \ p'xp' \le p'.$$

It remains to prove condition (AS). Every $g \in G(\mathcal{R})$ can be written as

$$g = pgp + p'gp' + pgp' + p'gp.$$

Put $g_s = pgp + p'gp'$, $g_a = pgp' + p'gp$. Then $U_p(g_s) = g_s$, while $U_p(g_a) = (2p-1)(pgp' + p'gp)(2p-1)$, and by direct computation we obtain that $U_p(g_a) = -g_a$. □

Theorem 4.5. *Let an effect ring $\mathcal{R}$ satisfy conditions of Theorem 4.4. Then $G(\mathcal{R})$ with $p \mapsto U_p$, $p \in P(\mathcal{R})$ is a symmetric group.*

Proof: We have to check conditions (i)–(iii) of Definition 3.4 Condition (i) follows from Theorem 4.4

(ii) Let $p, q \in P(\mathcal{R})$. Ten

$$U_p \circ U_q(g) = (2p-1)(2q-1)g(2q-1)(2p-1),$$
$$U_p(q) = (2p-1)q(2p-1) = 4pq - 2qp - 2pq + q,$$
$$(2U_p(q)-1) = 2(2p-1)q(2p-1) - 1 = (2p-1)(2q-1)(2p-1),$$

$$\begin{aligned}
&(2p-1)(2q-1)g(2q-1)(2p-1)\\
&= (2p-1)(2q-1)(2p-1)\\
&\quad \times[(2p-1)g(2p-1)](2p-1)(2q-1)(2p-1)\\
&= (2U_p(q)-1)(2p-1)g(2p-1)(2U_p(q)-1)\\
&\Rightarrow U_p \circ U_q(g) = U_{U_p(q)} \circ U_p(g) \ \forall g \in G(\mathcal{R}).
\end{aligned}$$

(iii) If $p \perp q$, then $U_p(q) = q$, hence by (ii), $U_p \circ U_q(g) = U_q \circ U_p(g)$, and since $p \perp q$ implies $pq = 0$, we obtain $(2p-1)(2q-1) = -2p - 2q + 1$, while $2(p+q) - 1 = 2p + 2q - 1$, so $U_p \circ U_q = U_{p+q}$. □

Theorem 4.6. *Let an effect ring $\mathcal{R}$ satisfy conditions of Theorem 4.4 and suppose that* 1 *is a generative unit in $G(\mathcal{R})$. Then every symmetry on $G(\mathcal{R})$, generated by an element $p \in G(\mathcal{R})$ and satisfying condition (AS), is of the form*

$$U_p(g) = (2p-1)g(2p-1)$$

where $p \in P(\mathcal{R})$.

Proof: Let U_p be a symmetry generated by p. By Lemma 3.3, $p \in P(\mathcal{R}) = P(E)$. For $x \in E$, x belongs to S_p iff $x = x_1 + x_2$ with $0 \leq x_1 \leq p, 0 \leq x_2 \leq p'$, equivalently, iff $x = pxp + p'xp'$. This implies that $S_p \cap E = \{x \in E : U_p(x) = x\} = \{x \in E : (2p-1)x(2p-1) = x\}$.

Take $g \in G(\mathcal{R})$ arbitrary. Then $g = g^+ - g^-$, $g^+, g^- \in G(\mathcal{R})^+$. Since 1 is generative, we may write $g^+ = \sum_{i=1}^n a_i$, $g^- = \sum_{j=1}^m b_j$, where $a_i, b_j \in E$ for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Now $pg^+p = \sum_{i=1}^n pa_ip$ and $p'g^+p' = \sum_{i=1}^m p'a_ip'$. Similar equalities, with a_i's replaced by b_j's, hold for g^-. It follows that $U_p((pg^+p) + (p'g^+p')) = pg^+p + p'g^+p' = (2p-1)(pg^+p + p'g^+p')(2p-1)$, and similarly, $U_p((pg^-p) + (p'g^-p')) = pg^-p + p'g^-p' = (2p-1)(pg^-p + p'g^-p')(2p-1)$, where the last equalities may be verified by direct computation.

We also have $pg^+p' = \sum_{i=1}^n pa_ip'$, $p'g^+p = \sum_{i=1}^n p'a_ip$ and $pg^-p' = \sum_{j=1}^m pb_jp'$, $p'g^-p = \sum_{j=1}^m p'b_jp$, which yields $U_p(pg^+p' + p'g^+p) = -pg^+p' - p'g^+p = (2p-1)(pg^+p' + p'g^+p)(2p-1)$ and $U_p(pg^-p' + p'g^-p) = -pg^-p' - p'g^-p = (2p-1)(pg^-p' + p'g^-p)(2p-1)$.

We can write $g = pgp + p'gp' + pgp' + p'gp$. Putting $g_s := pgp + p'gp'$, $g_a := pgp' + p'gp$, we have $g = g_s + g_a$, and from the previous part it follows that $U_p(g) = U_p(g_s) + U_p(g_a) = g_s - g_a = (2p-1)g(2p-1)$. □

5. SYMMETRIES AND COMPRESSIONS

Following definitions were introduced in Foulis (2004).

Definition 5.1. Let G be a partially ordered abelian group with order unit u and unit interval $E = [0, u]$. A mapping $J: G \to G$ is called a *retraction* on G iff:

(i) J is additive;
(ii) J is order preserving;
(iii) $J(u) \leq u$;
(iv) if $a \in G$ with $0 \leq a \leq J(u)$, then $J(a) = a$;
(v) J is idempotent.

Notice that if u is generative, then condition (v) in Definition 5.1 is redundant [(Foulis, 2004), Lemma 2.5]. If J is a retraction on G, then $J(u)$ is called the *focus* of J (Foulis, 2003). By [(Foulis, 2003), Lemma 2.3], if $p = J(u)$ is a focus of a retraction, then p is a principal element of E.

Definition 5.2. The retraction $J : G \to G$ is a *compression* if for all $k \in G$, $0 \leq k \leq u$, $J(k) = 0$ implies $g \leq u - J(u)$.

Let J and I be retractions on G. Then J and I are *quasicomplementary* iff for all $g \in G^+$, $J(g) = g \Leftrightarrow I(g) = 0$, and $J(g) = 0 \Leftrightarrow I(g) = g$. If J and I

are quasicomplementary, then J and I are compressions, and if p is the focus of J, the focus of I is $u - p$.

Let $\mathcal{R}$ be an effect ring, for every $p \in P(\mathcal{R})$, the mapping $J(g) = pgp$ is a retraction (Foulis, 2004). Moreover, $J'(g) = p'gp'$ is a retraction as well, and J and J' are quasicomplementary, hence they are compressions.

Theorem 5.3. (Foulis, 2004) [Theorem 3.6] *Let $\mathcal{R}$ be an archimedean effect ring, suppose that 1 is a generative order unit in $G(\mathcal{R})$, and let $J: G(R) \to G(R)$ be a retraction with $p := J(1)$. Then $p \in P(\mathcal{R})$ and $J(g) = pgp$ for all $g \in G(\mathcal{R})$.*

Putting together the statements of Theorem 4.6 and Theorem 5.3, we obtain the following results.

Theorem 5.4. *Let $\mathcal{R}$ be an effect ring satisfying conditions of Theorem 4.4 and suppose that 1 is a generative order unit in $G(\mathcal{R})$. Then $G(\mathcal{R})$ is a compressible group and every compression is of the form $J_p(g) = pgp$ for some $p \in P(\mathcal{R})$. For every $g \in G(\mathcal{R})$ and $p \in P(\mathcal{R})$, we have $g = J_p(g) + J_{p'}(g) + [g - J_p(g) - J_{p'}(g)]$. Defining $U_p(g) := J_p(g) + J_{p'}(g) - [g - J_p(g) - J_{p'}(g)]$, we obtain a symmetry with the stable part $g_s^p := J_p(g) + J_{p'}(g)$, and the alternating part $g_a^p := g - J_p(g) - J_{p'}(g)$. With the mapping $U((p, g): P(\mathcal{R}) \times G(\mathcal{R}) \to G(\mathcal{R})$, $G(\mathcal{R})$ becomes a symmetric group, where symmetries are given by $U_p(g) = (2p - 1)g(2p - 1)$.*

Conversely, consider $G(\mathcal{R})$ as a symmetric group with the symmetries $U_p(g) = (2p - 1)g(2p - 1)$, $p \in P(\mathcal{R})$. For every $e \in G(\mathcal{R})$, $0 \leq e \leq 1$, the stable part of e with respect to U_p is of the form $e_s^p = e_1 + e_2$, where $e_1 \leq p$, $e_2 \leq p'$. Then the mapping $J_p(e) := e_1$ can be extended to a compression on $G(\mathcal{R})$ of the form $J_p(g) = pgp$.

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